Information and Information Flow in Game Semantics

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Abstract. To successfully model an array of different programing languages, game semantics uses the detailed interactions between a system and its environment. To fine-tune such a model to a particular language, the set of strategies available to the players is limited using various conditions. In many cases these restrictions can be thought as the result of limiting the information available to the players to elaborate their strategies.
The aim of this paper is to study two frameworks to represent this partial knowledge explicitly. The first add to games as defined in game semantics equivalence relations identifying the position that cannot be distinguished by the players. The resulting structure is related to the process of restricting the player's strategies by a Galois connection.
The second framework is based upon a network game representation derived from coloured Petri nets. The games of the first framework are used to study the dynamics of this network game representation, so it can also be connected to game semantics. This relation is then used to give a characterisation of total information games in the network representation. A sketch of the basic constructions needed to define a category of games and strategies using the network representation is given.

1 Introduction

The successful application of the techniques of game semantics to construct fully abstract models for many different languages is due to the fact that one can restrict the set of strategies available to a player to those satisfying various conditions (innocence, bracketing, etc...) to fine-tune the model appropriately. Most of these restrictions implicitly limits the possible interactions between the players. Such limitations can be explained in two way. They can be thought as the result of removing moves in the games: this would leave the players with fewer options, and thus with fewer strategies. More interestingly, they could be seen as the result of the players having only limited knowledge of the state of the game when they decide upon a move to play in that state. In that case, the players also have fewer options because having less information makes them unable to distinguish some states of the game. This forces them to make one choice for each set of indistinguishable states instead of a choice for each state. This again limits the number of strategies available.

This paper present two ways to study the effects of limiting the options of the players explained as limited knowledge. The first one is an adaptation of the concepts of game theory used to study partial information to the formalism of game semantics. The second one use information flow to introduce partial knowledge in games.

2 Information structure of game semantics

2.1 The game semantics framework

We begin with a definition of game which keep only the structure relevant to study the implicit partial information introduced by limiting the player's strategies. The result is close to the basic definitions of [**?**], which are used to construct a category of games and strategies which models multiplicative linear logic. This abstracted concept is named "pregame" because it does not include any notion of payoff or winning strategy. It is also not assumed that there are two players that play alternately.

We denote the set of sequences (finite of infinite) of elements of a set A by A^∞, the prefix ordering of sequences by $\sqsubseteq$, the empty sequence is denoted by ϵ and the length of a finite sequence s by $|s|$. We use $s, t, \ldots$ for sequences, and $a, b, c, \ldots$ for moves, so when we write $sa \in A^\infty$, $s \in A^\infty$ and $a \in A$ are implicit.

Throughout this paper, P will denote a non-empty set of (possibly more than two) players. A pregame is formalised as the set of its possible histories or plays, each move being assigned to a player:

Definition 1. *A* pregame $\mathcal{P} = (M, H, N)$ *is a set* M *of* moves *together with a prefix-closed subset of* histories $H \subseteq M^\infty$ *containing* ϵ *and a function* $N : M \to P$ *that assigns each move to a player.*

This notion of pregame provide enough structure to define strategies:

Definition 2. *A* strategy *for the player* $p \in P$ *in* $\mathcal{P}$ *is a subset* $\sigma \subseteq H$ *such that*

1. $\epsilon \in \sigma$*,*
2. *if* $sa \in \sigma$*, then* $N(a) = p$*.*

The second condition of the above definition of strategy for p require that σ tells p how to choose moves as follows: if $sa \in \sigma$, then p chooses a on the position reached after the history s. Note is it not assumed that strategies are deterministic; a strategy σ is *deterministic* if $sa, sb \in \sigma \Rightarrow a = b$.

In game semantics, in order to get domains when we order strategies by inclusion, it is also assumed that every strategy σ is *prefix-closed*: $ta \sqsubseteq s \in \sigma$ and $N(a) = p$ implies that $ta \in \sigma$. In that case, if σ tells p how to decide at some point in an history s, then σ also gives moves to p for all previous points of s where p can move.

One can add other restrictions to strategies, such as being *history-free*:

$$sab \in \sigma, ta \in H \Rightarrow tab \in \sigma.$$

In "Hyland-Ong"-style game semantics (see [**?**] or chapter 3 of [**?**]), the set of histories H is the set of histories is defined using an enabling relation $\vdash$ between moves. It is assumed that if $a \vdash b$, then $N(a) \neq N(b)$, and that for all a with no enablers, $N(a) = \mathrm{O}$. The use of the enabling relation it to put restrictions on which moves can be made after a specific sequence of moves and specify which moves are considered as initial moves. Each occurrence of a non-initial move in an history is also equipped with a *justification pointer* which points to a previous occurrence of a move that enable it. Assuming $P = \{\mathrm{O}, \mathrm{P}\}$, where O ("Opponent") stands for the environment and P ("Player") for the system and that H contains only histories where the players alternates with O beginning, one can define inductively the P-*view* view(s) of an history s:

1. $\mathrm{V}(sa) = a$ if a is a an initial O-move,
2. $\mathrm{V}(satb) = \mathrm{V}(s)ab$ if $N(b) = \mathrm{O}$ and b points to a,
3. $\mathrm{V}(sa) = \mathrm{V}(s)a$ if $N(a) = \mathrm{P}$.

It is required that the justification pointers of non-initial occurrences of P-moves of any $s \in H$ point to a move of $\mathrm{V}(s)$; this condition is know as P-*visibility.*

In such a game, a strategy σ for P is said to be *innocent* if $sab \in \sigma$ and $t \in \sigma, ta \in H, \mathrm{V}(s) = \mathrm{V}(t)$, then $tab \in \sigma$. Note there is a unique way to add a justification pointer from b in tab such that $\mathrm{V}(tab) = \mathrm{V}(sab)$, so there is no ambiguity in the above condition.

The enabling relation used to define the set of histories also uses the extra moves labels ? and ! to mark moves that are "questions" and "answers", with extra conditions on $\vdash$ that requires that only questions can be initial moves or enablers. A *well-bracketed* strategy σ for P is such that if $sab \in \sigma$ and b is an answer, then the justification pointer of b points to the last occurrence of in the P-view a question by O which does not justify any answer (called the *pending question*).

2.2 Information structures

The tool used to study partial information in game theory is usually called the *extensive form* of a game. Basically, this represent the game as a tree of positions where the edges represent moves, plus a collection of equivalence relations I_p, one for each player p, which tell which positions are indistinguishable by p when p makes decisions. The equivalence classes of these relations are called *information sets.* Using extensive forms, one can represent concurrent moves and partial information by choosing appropriate information sets.

While a pregame can trivially be seen as a tree, there is a small difference between the underlying tree of an extensive form and a pregame: moves in a pregame do not correspond to the edges of its associated tree. In pregames, moves are actions that can be performed in many point of the game, while in extensive forms, the same action done at different points of the game will be considered as different moves. This does not cause any problems to define information sets in the formalism of pregames; in fact, the definition of information set is simplified.

For any $s \in H$, let $\text{next}_p(s) = \{a \in M | sa \in H \wedge N(a) = p\}$ be the set of moves available to p after s.

Definition 3. *An* information structure I *for a pregame* $\mathcal{P} = (M, H, N)$ *is a set of partial equivalence relations* I_p *on* H*, indexed by* $p \in P$*, such that*

$$sI_pt \Rightarrow \text{next}_p(s) = \text{next}_p(t)$$

and all $s \in H$ *with* $\text{next}_p(s) = \emptyset$ *are in the same* I_p*-equivalence class.*

The condition on I_p is necessary because $p \in P$ can always distinguish positions from which p cannot make the same moves. Note that we do not ask for the reciprocal condition to hold because it is possible to have positions where p have the same moves but where p have enough information to be able to distinguish them. It is also required that all positions where p does not have moves are identified by I_p. We call the equivalence classes of I_p *information sets* as in the case of extensive forms; the I_p information set containing an history s is denoted $I_p(s)$.

The concept of strategy must be adapted to take into account the potentially limited knowledge of the players when they make their decisions:

Definition 4. *A strategy* σ *for* $p \in P$ *is* compatible with I *if* sI_pt *and* $sa \in \sigma$ *imply that* $ta \in \sigma$*.*

This condition forces strategies for p to have the same set of choices of moves at every pair of positions that p cannot distinguish.

2.3 Allowed strategies and information structures

We can now connect the information structure view of partial knowledge to the way information is usually restricted in game semantics by allowing a player to use only a particular subset of all the possible strategies.

Let a *strategy collection* Σ be a function which gives for each $p \in P$ a set Σ_p of strategies for p. We order strategy collections by component-wise inclusion. Intuitively, the more choices the players have, the larger a strategy collection is and the more information they have.

We can also define a partial order on information structures for a pregame $\mathcal{P}$. Let I, J be two information structures on $\mathcal{P}$. We put $I \leq J$ if for all $p \in P$ and $s \in H$, $J_p(s) \subseteq I_p(s)$. Because fewer histories are identified, if $I \leq J$ the players have the same or less information if the information structure is J than if it is I. To illustrate the relationship between information structures and strategy collections, consider the simple pregame in figure 1 with players $P = \{\text{O}, \text{P}\}$, with two different information structures. To simplify, we consider in this example only deterministic prefix-closed strategies.

In the left case, the possible strategies for P are the following:

$$\{\epsilon\}, \{\epsilon, ac, bc\}, \{\epsilon, ad, bd\}.$$

Fig. 1. Two information structures on the same pregame. Bold edges are O-moves, normal edges are P-moves, and I_{O} and I_{P} information sets are also respectively drawn with bold and normal lines. The vertices not in any drawn information set for a player are assumed to be in the same one.

The player P has more possibilities in the right case, adding to the above list the two strategies

$$\{\epsilon, ac, bd\}, \{\epsilon, ad, bd\}.$$

We say a strategy collection Σ is compatible with an information structure I if every strategy in Σ is compatible with I. If a strategy collection is compatible an information structure I, then any smaller strategy collection will also be compatible with I. Let $G(I)$ be the strategy collection of all strategies compatible with the information structure I.

Similarly, given a strategy collection Σ for a pregame, there are many information structures that are compatible with all the strategies of Σ, the largest one, denoted by $F(\Sigma)$ being defined by

$$sF(\Sigma)_p t \iff \forall \sigma \in \Sigma_p . sa \in \sigma \leftrightarrow tb \in \sigma.$$

We can summarise the relationship between information structures and strategy collections by the following:

Proposition 1. *Let $\mathcal{P}$ be a pregame. There is a Galois connection between the partial order of information structures and the partial order of strategy collections.*

Proof. Both F and G are monotone. If $\Sigma \leq \Phi$ are two strategy collections, then since $\sigma \in \Sigma_p \Rightarrow \sigma \in \Phi$ we have that $\forall \sigma \in \Phi . sa \in \sigma \iff ta \in \sigma$ implies that $\forall \sigma \in \Sigma . sa \in \sigma \iff ta \in \sigma$, and therefore $F(\Sigma) \leq F(\Phi)$. If $I \leq J$ are two information structures, then $sJ_p t \Rightarrow sI_p t$. So if $\sigma \in G(I)$, then

$$sI_p t \Rightarrow (sa \in \sigma \iff ta \in \sigma)$$

and we may conclude that $\sigma \in G(J)$.

We now show that

$$F(\Sigma) \leq I \iff \Sigma \leq G(I).$$

Assuming $F(\Sigma) \leq I$, let $\sigma \in \Sigma_p$. If sI_pt and $sa \in \sigma$, then by hypothesis we also have $sF(\Sigma)_pt$, and therefore $ta \in \sigma$. We thus have that $\sigma \in G(I)$.

If instead we assume that $\Sigma \leq G(I)$, suppose sI_pt and let $\sigma \in \Sigma_p$. By hypothesis, σ is compatible with I, so we must have $sa \in \sigma \iff ta \in \sigma$. As this is the definition of $sF(\Sigma)_pt$, we have shown that $F(\Sigma) \leq I$.

Note that the proposition is true if we consider a specific player p: the relation between sets of strategies for p and the possible relations I_p, both ordered in the obvious way, is still a Galois connection if all else is remain fixed. This is illustrated using the pregame of figure 2, where all the possible strategies for P are history-free.

Fig. 2. An information structure for which all possible strategies for P are history free.

Proposition 1 gives us a way to characterise history-free strategies for P in terms of partial information: an *history-free* information structure for P is such that $saI_{\mathrm{P}}ta$ for all histories sa, ta with $\mathrm{next}_{\mathrm{P}}(sa) = \mathrm{next}_{\mathrm{P}}(sb) \neq \emptyset$. This force P to know only the last move and nothing of the rest of the history when making a decision.

A more important consequence of the last result is to allow to define when a particular restriction on strategies induce a natural limitation of the players information. If $\Sigma \leq \Phi$, $F(\Sigma) = F(\Phi)$ then the restriction of the strategy collection Φ to the strategy collection Σ cannot be considered as arising from a change of information structure. In that case, we have no choice but to consider the restriction as due to an implicit restriction of the pregame to a subset of H. An example of this situation is the restriction of the collection of all strategies to the collection of deterministic strategies or to well-bracketed strategies.

3 Representing games using information flow

The main drawback of the representation of games with partial information as pregames with information structures is that independent or concurrent moves must artificially appear sequentially in an history, using the information structure to encode the fact that when the second move is made, nothing is known about the result of the first move.

An alternate way to represent games which does not have this drawback is to use graphs representing how information is exchanged among the players. This approach as been used in [?] to produce an algorithm to find equilibrium strategies using symmetries of the games that could not be captured by an extensive form representation.

Note unfolded Petri nets are used in [?] to define "Petri games" used to interpret cut-free multiplicative linear logic proofs, but the interaction systems defined below use Petri nets to define games in a different way.

3.1 Interaction systems

An *interaction system* represent information flow among players using an evolution dynamics similar to the dynamics an *coloured Petri nets* where events are assigned to players (for a description of concept of coloured Petri net, one can see [?]). We begin by introducing the basic concepts of Petri net theory.

An *elementary net* $X = (V, E, \mathrm{In}, \mathrm{Out}, \Gamma_0)$ is a set of *places* V and a set of *events* E, disjoint from V, with two functions specifying the input and output sets $\mathrm{In}(a)$ and $\mathrm{Out}(a)$ of an event $a \in E$, plus a subset $\Gamma_0 \subseteq V$ called the *initial marking*. Events are thought as taking information in their input places and then putting information in their output places.

A set of places $\Gamma \subseteq X$ is called a *marking* of the elementary net; a marking is a set of places containing information at a given point of the evolution of the system. Given a marking Γ and an event a, we define $\Gamma^a = (\Gamma \setminus \mathrm{In}(a)) \cup \mathrm{Out}(a)$; this operation is allowed if an only if $\mathrm{In}(a) \subseteq \Gamma$ and $\mathrm{Out}(a) \cap (\Gamma \setminus \mathrm{In}(a)) = \emptyset$. We denote $\mathrm{next}(\Gamma)$ the set of $e \in E$ such that $\mathrm{In}(a) \subseteq \Gamma$. Given a sequence $s = a_1, \ldots, a_n$ of events, we denote by Γ^s the composition

$$(\cdots(\Gamma_1^a)\cdots)^{a_n}$$

provided all the involved operation are defined. If Γ_0^s is defined, we call s an *history*, and denote the set of histories by H. A *reachable marking* is a marking of the form Γ_0^s.

An elementary net X is *safe* if for all reachable marking Γ and $a \in \mathrm{next}(\Gamma)$, Γ^a is defined.

A *causality graph* is an elementary net X such that

1. All events are part of some history
2. All places are elements of some reachable marking
3. X is safe.

The first two condition make sure there is no unused places or unused events in the system. The safeness condition is necessary to be able to define the dynamics of interaction systems below.

Let a S family of sets indexed by elements of I. It is convenient to consider their Cartesian product $\times_i S(i)$ to be the set of functions $u : I \to \sum_i S(i)$ such that $u(i) \in S(i)$ (where $+$ is the disjoint union operation). Given any $J \subseteq I$, we put $S(J) = \times_{j \in J} S(j)$.

Definition 5. *Let P be a set of players. An* interaction system $\mathcal{X} = (X, S, F, N, u_0)$ *is a causality graph X with each event a assigned to a player $N(a)$, for each place x a set of states $S(x)$, for each event a function*

$$F_e : S(\mathrm{In}(a)) \to S(\mathrm{Out}(a)),$$

and finally an element $u_0 \in S(\Gamma_0)$.

When s is an history, an element $u \in S(\Gamma_0^s)$ is called a *state* on Γ^s . The dynamics on markings induce a dynamics on states, using the functions F_a associated to events to modify the data stored in the input places of a. Given a state u on a reachable marking Γ and $a \in \mathrm{next}(\Gamma)$, we define a new state u^a on Γ^a by

$$u^a(x) = \begin{cases} u^a(x) = \left[F_a(u|_{\mathrm{In}(a)})\right](x) \text{ if } x \in \mathrm{Out}(a), \\ u(x) \text{ if } x \notin \mathrm{Out}(a) \end{cases}$$

As it was done for markings, the definition of u^a can be extended to sequences of events.

Trace theory would provide another tool to describe the evolution of interaction systems. Since traces identify histories in which independent moves are permuted, they would give a better account of the evolution, one that does not force an unnatural sequentiality on concurrent moves in the system. The connection between elementary nets and trace structures is well known (for a description, see for example [**?**]), and it is not hard to adapt it to interaction systems. Nevertheless, since the goal of this paper is to study partial information and information flow in game semantics, using histories to describe the evolution has the advantage of a direct connection with pregames.

3.2 Strategies and information in interaction systems

In an interaction system, players interacts by sending information to each other, and make move by applying function to the information they receive. A player p can only see the part of its state which is on the input the p-moves when p makes a decision. To define a notion of strategy for p in an interaction system, states that p cannot be distinguish need to be identified.

Let $p \in P$ and $V_p = \{x \in V | \exists a \in E. x \in \mathrm{In}(a) \wedge N(a) = p\}$ be the set of all the places that p can see.

Definition 6. *Two reachable markings Γ, Δ of $\mathcal{X}$ are indistinguishable by p if*

$$\mathrm{next}_p(\Gamma) = \mathrm{next}_p(\Delta).$$

We denote this relation by $\Gamma \approx_p \Delta$.

Two states u *and* v *on* Γ *and* Δ *are indistinguishable by* p *if*

$$\Gamma \approx_p \Delta \text{ and } u|_{V_p} = v|_{V_p}.$$

We denote this relation by $u \sim_p v$.

The two relation $\approx_p$ and $\sim_p$ are partial equivalence relations defined only where p have moves. We denote the equivalence classes of a reachable marking Γ and a state u respectively by $[\Gamma]_p$ and $[u]_p$. Using these relations, the concept of strategy is defined as follows:

Definition 7. *Let* U *be the set of reachable states where* p *has moves. A* strategy *for* p *in an interaction system* $\mathcal{X}$ *is a partial function* σ *taking* $[u_0^s]_p \in U/\sim_p$ *to a subset of* $\text{next}_p(\Gamma_0^s)$.

At any given point, p is allowed to "look" at what is on the input of the moves and choose a set of moves $\sigma([u_0^s]_p)$. A deterministic strategy is one that alway pick singletons.

Note that the safeness condition of causality graphs have an effect on the concept of strategies. If many tokens were allowed to be in the same place, strategies would also have to specify to which one one must apply the function associated to the chosen. Safeness simplifies the concept of strategy for interaction systems and make it easier to connect it to strategies for pregame.

While it is not natural to define the concept of prefix closure in the setting of interaction systems because the order of concurrent moves should not be important, we can still use the histories to define the concept in order to be able to connect it to pregames prefix-closed strategies. A strategy σ for $p \in P$ is *prefix-closed* if each time it is defined at $[u_0^s]_p$ and $ta \sqsubseteq s$ with $N(a) = p$, it is also defined at $[u_0^t]_p$ and $a \in \sigma([u_0^t]_p)$.

3.3 Interaction systems and information structures

Given interaction system $\mathcal{X} = (X, S, F, N, u_0)$, the $\sim_p$ equivalence relations of the last section allows us to define easily an associated pregame $\mathcal{P} = (M, H, N)$ and an information structure. This is done as follows: M is the set of events E of X with the same associated players and H is the set of histories of $\mathcal{X}$. The information structure I is given by taking $I_p = \sim_p$.

Furthermore, a strategy σ for p in $\mathcal{X}$ can be converted into a strategy σ' for p in $\mathcal{P}$:

$$\sigma' = \{sa | s \in H, \sigma \text{ is defined at } [u_0^s]_p \text{ and } a \in \sigma([u_0^s]_p)\}$$

If σ is prefix-closed (resp. deterministic), then σ' is prefix-closed (resp. deterministic).

Consider for example the interaction system of figure 3. The set of states used are $\mathbb{B} = \{\mathbf{T}, \mathbf{F}\}$ and $\mathbf{Q} = \{\mathbf{q}\}$. The functions $\mathbf{Q}, \mathbf{T}, \mathbf{F}$ are the constant function returning respectively $\mathbf{q}, \mathbf{t}, \mathbf{f}$. All strategy for P correspond to innocent strategies of the associated pregame, if we add justifications pointers appropriately, since decisions are made using only the reachable states that can be seen by P. These correspond with the P-*views* in the game (**Bool** ⇒ **Bool**) ⇒ **Bool**.

Note that in the usual definition of Hyland-Ong games, one can have multiple game threads because nothing forbids to play a second initial move after a first one has been played. To recover this behaviour in the last example, one can play countably many copies of the interaction system to get a new one. If justification pointers added to the histories always points to move in the same copy, the strategies for P in the resulting game will automatically be *well-threaded*: the justification pointers for a P-move are pointing to a move of V(sa) (see for example chapter 3 of [**?**] for a more details about well-threaded strategies).

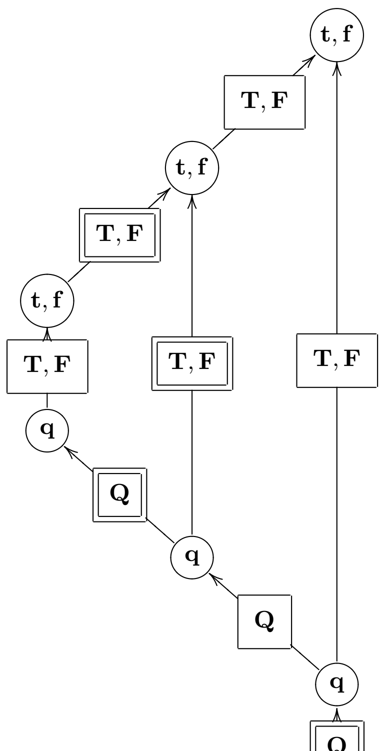

Fig. 3. An interaction system for innocent strategies in the game (**Bool** ⇒ **Bool**) ⇒ **Bool**. Places and set of events with same input and output sets are represented respectively as circles and boxes. A place x contains the various values of $S(x)$, and a box the functions F_a for each event a in it. Boxes drawn with single lines and double lines are respectively assigned to P and O.

3.4 Total information

A *total information* pregame $\mathcal{P}$ with information structure I is such that all equivalence classes $I_p(s)$ with $\text{next}_p(s) \neq \emptyset$ are singletons. An interaction system is of total information if its associated pregame and information structure is of total information.

An interesting problem is to characterise directly when an information system is of total information. There are intuitively two type of situations that prevent a player to know all about the past history of the game; they are illustrated by left and right nets of figure 4. In the left net a and b are two independent moves, so when P is about to play a, P has no information about whether b as occurred or not, and thus P cannot know everything about the state at that point. In the right net, take $S(x) = S(Y) = \{*\}$ and F_a, F_b are the function sending $*$ to $*$. In that case, P cannot infer from the state $u_0^a = u_0^b = *$ on $\{y\}$ whether a or b occurred. In general, all players p must be able to tell the last move and the previous state from the input state of p's moves. This is a local condition similar to injectivity. Before stating the main result of this section, we need to use

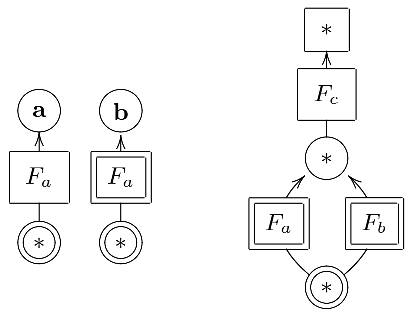

Fig. 4. Interactions systems where the global and local conditions fails

the underlying causality graph to formalise the intuitive independence relation between moves.

Definition 8. *Let* $\mathcal{X} = (X, S, F, N, u_0)$ *be an interaction system. The moves* $a, b \in E, a \neq b$ *are* independent *if there is a reachable marking* Γ *with* $\text{In}(a), \text{In}(b) \subseteq \Gamma$ *and* $\text{In}(a) \cap \text{In}(b) = \emptyset$.

We use the notation $a \| b$ to say that a and b are independent. We can now state the main result of this section:

Theorem 1. *An interaction system is of total information if and only if*

1. *(Global condition) There is no pair of independent moves,*

2. *(Local condition) If* $u_0^s \sim_p u_0^t$, $\mathrm{next}_p(u_0^s) = \mathrm{next}_p(u_0^t) \neq \emptyset$ *and* $s = \epsilon$ *or* $t = \epsilon$, *then* $s = t = \epsilon$. *If* $u_0^{sa} \sim_p u_0^{tb}$, $\mathrm{next}_p(u_0^{sa}) = \mathrm{next}_p(u_0^{tb}) \neq \emptyset$ *and* $a \not\parallel b$, *then* $a = b$ *and* $u_0^s \sim_{N(a)} u_0^t$.

Proof. An interaction system is of total information if and only if $u_0^s \sim_p u_0^t$ and $\mathrm{next}_p(s) = \mathrm{next}_p(t) \neq \emptyset$ implies $s = t$.

To show that the global and local conditions suffice, we proceed by induction on $|s|$. If $s = \epsilon$, then the local condition implies that $t = \epsilon$.

If $s = s'a$ and $u_0^{s'a} \sim_p u_0^t$, then the local condition forces $t \neq \epsilon$. So we must have $u_0^{s'a} \sim_p u_0^{t'a}$ for some decomposition $t = t'b$. By the global condition, $a \not\parallel b$, so the local condition implies that $a = b$ and $u_0^{s'} \sim_{N(a)} u_0^{t'}$. Since $\mathrm{next}_{N(a)} \neq \emptyset$, by induction hypothesis we have that $s' = t'$ and thus $s = t$.

We now show the necessity of the global and local conditions. Assume that the global condition does not hold, i.e. that there is a pair of distinct moves a, b with $a \parallel b$. By definition, there is an history s such that $u_0^{sab} = u_0^{sba}$. If there is a $p \in P$ with $\mathrm{next}_p(sab) = \mathrm{next}_p(sba) \neq \emptyset$ and the interaction system is of total information, we must conclude that $sa = sb$, which is a contradiction.

If it is the local condition which does not hold, there must be $p \in P$ and histories sa, tb with $\mathrm{next}_p(u_0^{sa}) = \mathrm{next}_p(u_0^{tb}) \neq \emptyset$, $u_0^{sa} \sim_p u_0^{tb}$ and $a \not\parallel b$, but also such that $a \neq b$ or $u_0^s \not\sim_{N(a)} u_0^t$. If the interaction system is of total information, then $u_0^{sa} \sim_p u_0^{tb}$ implies that $sa = tb$, which contradicts $a \neq b$ or $u_0^s \sim_{N(a)} u_0^t$. Again, the interaction system cannot be of total information, and it is thus the case as soon as either the local or the global condition are false.

4 Toward a category of interaction systems and strategies

This last section is a sketch of the basic constructions needed for game semantics done using interaction systems.

4.1 Rules

In interaction systems, nothing specifies who is to play after a sequence of move has occurred. While this is not problematic in the previous sections, in the game semantics constructions it is necessary to have this resolved. We use the following concept:

Definition 9. *Let* $\mathcal{X} = (X, S, F, N, u_0)$ *be an interaction system. A* rule R *for* $\mathcal{X}$ *is a function taking each history to a player* $p \in P$.

For example, in the rule used the most often in game semantics, both players can have moves in a given position, but there is usually a rule saying the player must play alternately, with the "environment" player beginning.

This formulation induces a natural question about the standard hypothesis of game semantics: why is working with two players - and not more - playing alternately so important in game semantics? Relaxing this rule leads to associativity problems (as mentioned with respect to Blass games in [?]). Generalisations to more than two players have not been used, even if such a generalisation should intuitively be related to interactions between multiple agents described by a concurrent language (for example, a game semantics for such a language with using the interactions of two players is given in [?]). In that case, there are two problems: first, there is no natural replacement for the basic two player game operation of inverting the players roles, and second, there is no natural generalisation of the alternation rule.

4.2 Operation on interaction systems with rules

We now follow the lines of the usual game semantics construction of a symmetric monoidal category of game and strategies. At this point, we need to take $P = \{\mathrm{O}, \mathrm{P}\}$. We also only consider interaction systems equipped with rules R of the type used in game semantics: O starts and players alternate afterward. We abuse the notation and will denote all these rules by R, no matter which interaction system they are associated to.

Let $\mathcal{X} = (X, S^X, M^X, N^X, u_0), \mathcal{Y} = (Y, S^Y, N^Y, v_0)$ be two interaction systems and J, K be their respective set of reachable markings. We define a new causality graph $X \odot Y$ by $V(X \odot Y) = V(X) \times V(Y)$,

$$E(X \odot Y) = (E(X) \times K) + (J \times E(Y)),$$

and finally

$$\mathrm{In}((a, \Gamma)) = \mathrm{In}(a) \times \Gamma$$

and similarly for $\mathrm{Out}(a\Gamma)$ and for events of the form (Φ, b). Using the data defining $\mathcal{X}$ and $\mathcal{Y}$, we define a new interaction system $\mathcal{X} \odot \mathcal{Y} = (X \odot Y, S, M, N, w_0)$:

- $S((x, y)) = S^X(x) \times S^Y(y)$, i.e. states in the new interaction systems are pairs of states from each system,
- for $(a, \Gamma) \in E(X)$, we take $M((a, \Gamma))$ to be the function that for each $y \in \Gamma$ acts on the part of the state on $\mathrm{In}(a) \times \{y\}$ as $M^X(a) \times \mathrm{Id}_{S(y)}$, and similarly for events of the form (Φ, b),
- $N((a, \Gamma)) = N^X(a)$, and similarly for events of the form (Φ, b), i.e. each move stay associated to the same player,
- For each x and y in the respective initial markings of X and Y, $w_0(x, y) = (u_0(x), v_0(y))$.

Given an interaction system $\mathcal{X}$, let $\mathcal{X}|_R$ be the result of removing all moves a from $\mathcal{X}$ that are inconsistent with R in the sense that there are histories s for which $a \in \mathrm{next}_p(s)$ but $R(s) \neq p$.

The new interaction system $\mathcal{X} \otimes \mathcal{Y}$ is defined to be $(\mathcal{X} \odot \mathcal{Y})|_R$. The operation $\mathcal{X} \multimap \mathcal{Y}$ can be defined to be $(\mathcal{X}^\perp \otimes \mathcal{Y})|_R$, where $\mathcal{X}^\perp$ is $\mathcal{X}$ with inverted player roles.

Taking the above definitions as a starting point, one face a problem when defining identity strategies and composition. The core of this problem is that in both constructions P must know what the last move is and in which component it was played. Therefore, in order to define these constructions, one can restrict the kind of interaction systems considered to those where P can always determine the last move:

$$u_0^{sa} \sim_{\mathrm{P}} u_0^{tb} \Rightarrow a = b.$$

5 Conclusion

We present in this paper to possible frameworks to study how partial information can arise by limiting the player's strategies in a game. Interesting developpements are possible in both cases.

The framework of pregames and information structures can be used to define the basic constructions of game semantics, taking the information relations I of the tensor of two pregames $\mathcal{P}, \mathcal{Q}$ with information structures J, K to be related by

$$sI_pt \iff s_{\mathcal{P}}J_pt_{\mathcal{P}} \wedge s_{\mathcal{Q}}K_pt_{\mathcal{Q}},$$

where $s_{\mathcal{P}}$ is the sequence obtained by keeping only the moves of s that are in $\mathcal{P}$. Some of the various lemmas that say that a certain type of strategies is closed under composition can be reformulated by saying, in the notation used above, the composition of strategies σ, τ respectively compatible with J and K are compatible with I. This is the case for innocent strategies for example. It is reasonable to think there is a general result of this sort that can be proved using information structures.

The preliminary theory of interaction systems presented here provide a new way to look at the interactions between the system and its environment, and a detailed study of its many possible connections to game semantics is still in progress.